DOLBEAULT COHOMOLOGY OF A LOOP SPACE

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Abstract. The loop space $L\mathbb{P}_1$ of the Riemann sphere is an infinite dimensional complex manifold consisting of maps (loops) $S^1 \to \mathbb{P}_1$ in some fixed C^k or Sobolev $W^{k,p}$ space. In this paper we compute the Dolbeault cohomology groups $H^{0,1}(L\mathbb{P}_1)$.

0. Introduction

Loop spaces LM of compact complex manifolds M promise to have rich analytic cohomology theories, and it is expected that sheaf and Dolbeault cohomology groups of LM will shed new light on the complex geometry and analysis of M itself. This idea first occurs in [W], in the context of the infinite dimensional Dirac operator, and then in [HBJ] that touches upon Dolbeault groups of loop spaces; but in all this both works stay heuristic. Our goal here is rigorously to compute the $H^{0,1}$ Dolbeault group of the first interesting loop space, that of the Riemann sphere $\mathbb{P}_1$. The consideration of $H^{0,1}(L\mathbb{P}_1)$ was directly motivated by [MZ], that among other things features a curious line bundle on $L\mathbb{P}_1$. More recently, the second named author in [Z] classified all holomorphic line bundles on $L\mathbb{P}_1$ that are invariant under a certain group of holomorphic automorphisms of $L\mathbb{P}_1$—a problem closely related to describing (a certain subspace of) $H^{0,1}(L\mathbb{P}_1)$. One noteworthy fact that emerges from the present research is that analytic cohomology of loop spaces, unlike topological cohomology (cf. [P, Theorem 13.14]), is rather sensitive to the regularity of loops admitted in the space. Another concerns local functionals, a notion from theoretical physics. Roughly, if M is a manifold, a local functional on a space of loops $x\colon S^1 \to M$ is one of form

$$f(x) = \int_{S^1} \Phi(t, x(t), \dot{x}(t), \ddot{x}(t), \dots)dt,$$

where Φ is a function on $S^1\times$ an appropriate jet bundle of M. It turns out that all cohomology classes in $H^{0,1}(L\mathbb{P}_1)$ are given by local functionals. Nonlocal cohomology classes exist only perturbatively, i.e., in a neighborhood of constant loops in $L\mathbb{P}_1$; but none of them extends to the whole of $L\mathbb{P}_1$.

We fix a smoothness class C^k, $k = 1, 2, \dots, \infty$, or Sobolev $W^{k,p}$, $k = 1, 2, \dots, 1 \le p < \infty$. If M is a finite dimensional complex manifold, consider the space $LM = L_kM$, resp. $L_{k,p}M$ of maps $S^1 = \mathbb{R}/\mathbb{Z} \to M$ of the given regularity. These spaces are complex manifolds modeled on a Banach space, except for $L_\infty M$, which is modeled on a Fréchet space. We shall focus on the loop space(s) $L\mathbb{P}_1$. As on any complex manifold, one can consider the

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space $C^\infty_{r,q}(L\mathbb{P}_1)$ of smooth (r,q) forms, the operators $\bar\partial : C^\infty_{r,q}(L\mathbb{P}_1) \to C^\infty_{r,q+1}(L\mathbb{P}_1)$, and the associated Dolbeault groups $H^{r,q}(L\mathbb{P}_1)$; for all this, see e.g. [L1,2]. On the other hand, let $\mathfrak{F}$ be the space of holomorphic functions $F\colon \mathbb{C}\times L\mathbb{C}\to\mathbb{C}$ that have the following properties:

(1) $F(\zeta/\lambda, \lambda^2 y) = O(\lambda^2)$, as $\mathbb{C}\ni\lambda\to 0$;
(2) $F(\zeta, x+y) = F(\zeta,x)+F(\zeta,y)$, if $\operatorname{supp} x\cap\operatorname{supp} y=\emptyset$;
(3) $F(\zeta, y+\text{const}) = F(\zeta,y)$.

As we shall see, the additivity property (2) implies $F(\zeta,y)$ is local in y.

Theorem 0.1. *$H^{0,1}(L\mathbb{P}_1)\approx\mathbb{C}\oplus\mathfrak{F}$.*

In the case of $L_\infty\mathbb{P}_1$, examples of $F\in\mathfrak{F}$ are

$$F(\zeta,y)=\zeta^\nu\Big\langle \Phi,\ \prod_{j=0}^m y^{(d_j)}\Big\rangle, \tag{0.1}$$

where Φ is a distribution on $S^1, y^{(d)}$ denotes d'th derivative, each $d_j\ge d_0=1$, and $0\le\nu\le 2m$. A general function in $\mathfrak{F}$ can be approximated by linear combinations of functions of form (0.1), see Theorem 1.5.

On any, possibly infinite dimensional complex manifold X the space $C^\infty_{r,q}(X)$ can be given the compact–C^∞ topology as follows. First, the compact–open topology on $C^\infty_{0,0}(X) = C^\infty(X)$ is generated by C^0 seminorms $\|f\|_K=\sup_K|f|$ for all $K\subset X$ compact. The family of C^ν seminorms is defined inductively: each $C^{\nu-1}$ seminorm $\|\ \ \|$ on $C^\infty(TX)$ induces a C^ν seminorm $\|f\|'=\|df\|$ on $C^\infty(X)$. The collection of all C^ν seminorms, $\nu=0,1,\dots$, defines the compact–C^∞ topology on $C^\infty(X)$. The compact–C^∞ topology on a general $C^\infty_{r,q}(X)$ is induced by the embedding $C^\infty_{r,q}(X)\subset C^\infty(\overset{r+q}{\oplus} TX)$. With this topology $C^\infty_{r,q}(X)$ is a separated locally convex vector space, complete if X is first countable. The quotient space $H^{r,q}(X)$ inherits a locally convex topology, not necessarily separated. We note that on the subspace $\mathcal{O}(X)\subset C^\infty(X)$ of holomorphic functions the compact–C^∞ topology restricts to the compact–open topology. The isomorphism in Theorem 0.1 is topological; it is also equivariant with respect to the obvious actions of the group of C^k diffeomorphisms of S^1.

There is another group, the group $G\approx\mathrm{PSL}(2,\mathbb{C})$ of holomorphic automorphisms of $\mathbb{P}_1$, whose holomorphic action on $L\mathbb{P}_1$ (by post–composition) and on $H^{0,1}(L\mathbb{P}_1)$ will be of greater concern to us. Theorems 0.2, 0.3, 0.4 below will describe the structure of $H^{0,1}(L\mathbb{P}_1)$ as a G–module. Recall that any irreducible (always holomorphic) G–module is isomorphic, for some $n=0,1,\dots$, to the space $\mathfrak{K}_n$ of holomorphic differentials $\psi(\zeta)(d\zeta)^{-n}$ of order $-n$ on $\mathbb{P}_1$; here ψ is a polynomial, deg $\psi\le 2n$, and G acts by pullback. (For this, see [BD, pp. 84-86], and note that the subgroup $\approx\mathrm{SO}(3)$ formed by $g\in G$ that preserve the Fubini–Study metric is a maximally real submanifold; hence the holomorphic representation theory of G agrees with the representation theory of $\mathrm{SO}(3)$.) The n'th isotypical subspace of a G–module V is the sum of all irreducible submodules isomorphic to $\mathfrak{K}_n$. In particular, the 0'th isotypical subspace is the space V^G of fixed vectors.

Theorem 0.2. *If $n \geq 1$, the n'th isotypical subspace of $H^{0,1}(L_\infty \mathbb{P}_1)$ is isomorphic to the space $\mathfrak{F}^n$ spanned by functions of form (0.1), with $m = n$.*

The isomorphism above is that of locally convex spaces, as $\mathfrak{F}$ or $\mathfrak{F}^n$ have not been endowed with an action of G yet. But in Section 2 they will be, and we shall see that the isomorphism in question is a G–morphism.—The fixed subspace of $H^{0,1}(L\mathbb{P}_1)$ can be described more explicitly, for any loop space:

Theorem 0.3. *The space $H^{0,1}(L\mathbb{P}_1)^G$ is isomorphic to $C^{k-1}(S^1)^*$, resp. $W^{k-1,p}(S^1)^*$, if the dual spaces are endowed with the compact–open topology.*

The isomorphisms in Theorem 0.3 are not Diff S^1 equivariant. To remedy this, one is led to introduce the spaces $C^l_r(S^1)$, resp. $W^{l,p}_r(S^1)$ of differentials $y(t)(dt)^r$ of order r on S^1, of the corresponding regularity; $L^p_r = W^{0,p}_r$. Then $H^{0,1}(L\mathbb{P}_1)^G$ will be Diff S^1 equivariantly isomorphic to $C^{k-1}_1(S^1)^*$, resp. $W^{k-1,p}_1(S^1)^*$.

For low regularity loop spaces one can very concretely represent all of $H^{0,1}(L\mathbb{P}_1)$:

Theorem 0.4. *(a) If $1 \leq p < 2$, all of $H^{0,1}(L_{1,p}\mathbb{P}_1)$ is fixed by G, hence it is isomorphic to $L^{p'}(S^1)$, with $p' = p/(p-1)$.*

(b) If $1 \leq p < \infty$ then $H^{0,1}(L_{1,p}\mathbb{P}_1)$ is isomorphic to

$$\bigoplus_{0 \leq n \leq p-1} \mathfrak{K}_n \otimes L^{p/(n+1)}_{n+1}(S^1)^* \approx \bigoplus_{0 \leq n \leq p-1} \mathfrak{K}_n \otimes L^{p_n}_{-n}(S^1), \qquad p_n = \frac{p}{p-1-n},$$

and so it is the sum of its first $[p]$ isotypical subspaces. Indeed, the isomorphisms above are $G \times$ Diff S^1 equivariant, G, resp. Diff S^1 acting on one of the factors $\mathfrak{K}_n$, L^q_r naturally, and trivially on the other.

Again, the dual spaces are endowed with the compact–open topology.

It follows that the infinite dimensional space $H^{0,1}(L_{1,p}\mathbb{P}_1)$ can be understood in finite terms, if it is considered as a representation space of S^1. Here S^1 acts on itself (by translations), hence also on $L\mathbb{P}_1$ and on $H^{0,1}(L\mathbb{P}_1)$. One can read off from Theorem 0.4 that each irreducible representation of S^1 occurs in $H^{0,1}(L_{1,p}\mathbb{P}_1)$ with the same multiplicity $[p]^2$. On the other hand, for spaces of loops of regularity at least C^1, in $H^{0,1}(L\mathbb{P}_1)$ each irreducible representation of S^1 occurs with infinite multiplicity and, somewhat contrary to earlier expectations, it is not possible to associate with this cohomology space even a formal character of S^1. This indicates that Dolbeault groups of general loop spaces LM should be studied as representations of Diff S^1 rather than S^1.

The structure of this paper is as follows. In Sections 1 and 2 we study the space $\mathfrak{F}$ as a G–module. We connect it with a similar but simpler space of functions that are required to satisfy only the first two of the three conditions defining $\mathfrak{F}$ (Theorem 1.1). Theorem 1.1 will be needed in proving the isomorphism $H^{0,1}(L\mathbb{P}_1) \approx \mathbb{C} \oplus \mathfrak{F}$, and also in concretely representing elements of $\mathfrak{F}$. Further, we shall rely on Theorem 1.1 in identifying isotypical

subspaces of $\mathfrak{F}$ (Theorems 2.1, 2.2). This will then prove Theorems 0.2, 0.3, and 0.4, modulo Theorem 0.1.

In Section 3 we introduce a G–module $\mathfrak{H}$ of holomorphic Čech cocycles of $L\mathbb{P}_1$, and prove $H^{0,1}(L\mathbb{P}_1) \approx \mathfrak{H}$ (Theorem 3.3). In Section 4 we construct a morphism $\alpha\colon \mathfrak{H} \to \mathfrak{F}$ that, in Section 5, is shown to induce an isomorphism $\mathfrak{H}/\mathrm{Ker}\ \alpha \to \mathfrak{F}$. Also, $\dim \mathrm{Ker}\ \alpha = 1$ (Theorem 5.1). Finally, in Section 6 we show how all this, put together, proves the results formulated in this introduction.

1. The Space $\mathfrak{F}$

In this Section and the next we shall study the structure of the space $\mathfrak{F}$, independently of any cohomological content. It will be convenient to allow (but only in this Section!) k to be any integer; when $k < 0$, elements of $C^k(S^1)$, $W^{k,p}(S^1)$ are distributions, locally equal to the $-k$'th derivative of functions in $C(S^1)$, $L^p(S^1)$. Let $L^-\mathbb{C}$ denote the space $C^{k-1}(S^1)$, resp. $W^{k-1,p}(S^1)$. We shall write $L^{(-)}\mathbb{C}$ to mean either $L\mathbb{C}$ or $L^-\mathbb{C}$. Consider the space $\tilde{\mathfrak{F}}$ of those $F \in \mathcal{O}(\mathbb{C} \times L^-\mathbb{C})$ that have properties (1) and (2) of the Introduction. We shall refer to (2) as additivity. A function $F \in \mathcal{O}(\mathbb{C} \times L^{(-)}\mathbb{C})$ will be said to be posthomogeneous of degree m if $F(\zeta, \cdot)$ is homogeneous of degree m for all $\zeta \in \mathbb{C}$. Posthomogeneous degree endows the spaces $\mathfrak{F}$ and $\tilde{\mathfrak{F}}$ with a grading.—All maps below, unless otherwise mentioned, will be continuous and linear.

Theorem 1.1. *The graded linear map $\tilde{\mathfrak{F}} \ni \tilde{F} \mapsto F \in \mathfrak{F}$ given by $F(\zeta, y) = \tilde{F}(\zeta, \dot{y})$ has a graded right inverse, and its kernel consists of functions $\tilde{F}(\zeta, x) = const \int_{S^1} x$.*

First we shall consider functions $E \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, that are independent of ζ. We denote the space of these functions $\mathfrak{E} \subset \mathcal{O}(L\mathbb{C})$, resp. $\tilde{\mathfrak{E}} \subset \mathcal{O}(L^-\mathbb{C})$, graded by degree of homogeneity. Additivity of $E \in \mathcal{O}(L^{(-)}\mathbb{C})$ implies $E(0) = 0$, which in turn implies property (1) of the Introduction. Let

$$E = \sum_1^\infty E_m, \qquad E_m(y) = \int_0^1 E(e^{2\pi i\tau}y)e^{-2m\pi i\tau}d\tau \tag{1.1}$$

be the homogeneous expansion of a general $E \in \mathcal{O}(L^{(-)}\mathbb{C})$ vanishing at 0. Consider tensor powers $(L^{(-)}\mathbb{C})^{\otimes m}$ of the vector spaces $L^{(-)}\mathbb{C}$ over $\mathbb{C}$. In particular, $C^\infty(S^1)^{\otimes m}$ is an algebra, and a general $(L^{(-)}\mathbb{C})^{\otimes m}$ is a module over it. Each E_m in (1.1) induces a symmetric linear map

$$\mathcal{E}_m\colon (L^{(-)}\mathbb{C})^{\otimes m} \to \mathbb{C},$$

called the polarization of E_m. On monomials $\mathcal{E}_m$ is defined by

$$\mathcal{E}_m(y_1 \otimes \ldots \otimes y_m) = \frac{1}{2^m m!} \sum_{\epsilon_j = \pm 1} \epsilon_1 \ldots \epsilon_m E_m(\epsilon_1 y_1 + \ldots + \epsilon_m y_m), \tag{1.2}$$

and then extended by linearity. Thus $E_m(y) = \mathcal{E}_m(y^{\otimes m})$.—We shall call $w \in (L^{(-)}\mathbb{C})^{\otimes m}$ degenerate if it is a linear combination of monomials $y_1 \otimes \ldots \otimes y_m$ with one $y_j = 1$.

Lemma 1.2. *(a) E is additive if and only if $\mathcal{E}_m(y_1 \otimes \ldots \otimes y_m) = 0$ whenever $\bigcap_1^m \operatorname{supp} y_j = \emptyset$.*

(b) $E(y + \mathrm{const}) = E(y)$ if and only if $\mathcal{E}_m(w) = 0$ whenever w is degenerate.

Proof. (a) Clearly E is additive precisely when all the E_m are, whence it suffices to prove the claim when E itself is homogeneous, of degree m, say. In this case $\mathcal{E}_n = 0,\ n \neq m$. Denoting $\mathcal{E}_m$ by $\mathcal{E}$, it is also clear that the condition on $\mathcal{E}$ implies E is additive. We show the converse by induction on m, the case $m = 1$ being obvious. Let $x, y \in L^{(-)}\mathbb{C}$ have disjoint support, so that

$$\mathcal{E}((x+y)^{\otimes m}) = \mathcal{E}(x^{\otimes m}) + \mathcal{E}(y^{\otimes m}). \tag{1.3}$$

Write λx for x and separate terms of different degrees in λ to find $\mathcal{E}(x \otimes \ldots \otimes y) = 0$, which settles the case $m = 2$. Now if the case $m - 1 \geq 2$ is already settled, take a $z \in L^{(-)}\mathbb{C}$ with $\operatorname{supp} y \cap \operatorname{supp} z = \emptyset$, and write $x + \lambda z$ for x in (1.3). Considering the terms linear in λ we obtain

$$\mathcal{E}(z \otimes (x+y)^{\otimes m-1}) = \mathcal{E}(z \otimes x^{m-1}) + \mathcal{E}(z \otimes y^{m-1}), \tag{1.4}$$

the last term being 0. The same will hold if $\operatorname{supp} x \cap \operatorname{supp} z = \emptyset$. Since any $z \in L^{(-)}\mathbb{C}$ can be written $z' + z''$ with the support of z' (resp. z'') disjoint from the support of x (resp. y), (1.4) in fact holds for all z. By the induction hypothesis applied to $\mathcal{E}(z \otimes \cdot)$

$$\mathcal{E}(z \otimes y_2 \otimes \ldots \otimes y_m) = 0, \qquad \text{if} \quad \bigcap_2^m \operatorname{supp}\, y_j = \emptyset.$$

Suppose now $\bigcap_1^m \operatorname{supp} y_j = \emptyset$ and write $y_1 = y' + y''$ with $y' = 0$ near $\bigcap_{j\neq 2} \operatorname{supp} y_j$ and $y'' = 0$ near $\bigcap_{j \neq 3} \operatorname{supp} y_j$. Then

$$\mathcal{E}(y_1 \otimes \ldots \otimes y_m) = \mathcal{E}(y' \otimes \ldots \otimes y_m) + \mathcal{E}(y'' \otimes \ldots \otimes y_m) = 0.$$

(b) Again we assume E is m–homogeneous, and again one implication is trivial. So assume $\mathcal{E}((y+1)^{\otimes m}) = \mathcal{E}(y^{\otimes m})$, where $\mathcal{E} = \mathcal{E}_m$. Differentiating both sides in the directions $y_2, \ldots, y_m$ and setting $y = 0$ we obtain $\mathcal{E}(1 \otimes y_2 \otimes \ldots \otimes y_m) = 0$, whence the claim follows.

Proposition 1.3. *The graded map $\tilde{\mathfrak{E}} \ni \tilde{E} \mapsto E \in \mathfrak{E}$ given by $E(y) = \tilde{E}(\dot{y})$ has a graded right inverse, and its kernel is spanned by $\tilde{E}(x) = \int_{S^1} x$.*

We shall write $\int x$ for $\int_{S^1} x$.

Proof. (a) To identify the kernel, because of homogeneous expansions, it will suffice to deal with homogeneous $\tilde{E}$. So assume $\tilde{E} \in \tilde{\mathfrak{E}}$ is homogeneous of degree m and $\tilde{E}(\dot{y}) = 0$ for all $y \in L\mathbb{C}$. Its polarization $\tilde{\mathcal{E}}$ satisfies $\tilde{\mathcal{E}}(\dot{y}_1 \otimes \ldots \otimes \dot{y}_m) = 0$. If $m = 1$, this implies

$\tilde{E}(x) = \text{const} \int x$, so from now on we assume $m \geq 2$, and first we prove by induction that $\tilde{\mathcal{E}}(x_1 \otimes \ldots \otimes x_m) = \text{const} \prod \int x_j$. Suppose we already know this for $m-1$. Then

$$\tilde{\mathcal{E}}(\dot{y} \otimes x_2 \otimes \ldots \otimes x_m) = c(\dot{y}) \prod_2^m \int x_j.$$

With arbitrary $x_1 \in L^-\mathbb{C}$ the function $x_1 - \int x_1$ is of form $\dot{y}$, so $x_1 = \dot{y} + \int x_1$, and

$$\tilde{\mathcal{E}}(x_1 \otimes \ldots \otimes x_m) = l(x_1) \prod_2^m \int x_j + \tilde{\mathcal{E}}(1 \otimes x_2 \otimes \ldots \otimes x_m) \int x_1, \tag{1.5}$$

where $l(x_1) = c(x_1 - \int x_1)$ is linear in x_1. If $\int x_1 = 0$ and $\operatorname{supp} x_1 \neq S^1$, then we can choose $x_2, \ldots$ so that $\bigcap_1^m \operatorname{supp} x_j = \emptyset$ but $\int x_j \neq 0,\ j \geq 2$. This makes the left hand side of (1.5) vanish by Lemma 1.2a, and gives $l(x_1) = 0$. Since any $x_1 \in L^-\mathbb{C}$ with $\int x_1 = 0$ can be written $x_1 = x' + x''$ with $\int x' = \int x'' = 0$ and $\operatorname{supp} x', \operatorname{supp} x'' \neq S^1$, it follows that $l(x_1) = 0$ whenever $\int x_1 = 0$. Hence $l(x_1) = \text{const} \int x_1$. In particular, the first term on the right of (1.5) is symmetric in x_j. Therefore the second term must be symmetric too, which implies this term is const $\prod_1^m \int x_j$. Thus $\tilde{E}(x) = \text{const}(\int x)^m$.

Yet for $m \geq 2$ $\tilde{E}(x) = \text{const}(\int x)^m$ is additive only if it is identically zero; so that indeed $\tilde{E}(x) = \text{const} \int x$, as claimed.

(b) To construct the right inverse, consider $E \in \mathfrak{E}$ with homogeneous expansion (1.1). We shall construct m–homogeneous polynomials $\tilde{E}_m \in \tilde{\mathfrak{E}}$ such that $E_m(y) = \tilde{E}_m(\dot{y})$; the case $m = 1$ being obvious, we assume $m \geq 2$. Let us say that an n–tuple of functions $\rho_\nu \colon S^1 \to \mathbb{C}$ is centered if $\bigcap_1^n \operatorname{supp} \rho_\nu \neq \emptyset$. We start by fixing a C^∞ partition of unity $\sum_{\rho \in P} \rho = 1$ on S^1 such that each $\operatorname{supp} \rho$ is an arc of length $< 1/4$. This implies that $\bigcup_1^n \operatorname{supp} \rho_\nu$ is an arc of length $< 1/2$ if $\rho_1, \ldots, \rho_n \in P$ are centered. Given $x \in L^-\mathbb{C}$, for each centered $R = (\rho_1, \ldots, \rho_n)$ in P construct $y_R \in L\mathbb{C}$ so that $\dot{y}_R = x$ on a neighborhood of $\bigcup_1^n \operatorname{supp} \rho_\nu$, making sure that $y_R = y_Q$ if Q and R agree as sets. For noncentered n–tuples R in P let $y_R \in L\mathbb{C}$ be arbitrary. We shall refer to the y_R as local integrals.

If Q, R are centered tuples in P then

$$y_Q - y_R = c_{QR} = \text{constant} \qquad \text{on} \quad \Big(\bigcup_{\rho \in Q} \operatorname{supp} \rho\Big) \cap \Big(\bigcup_{\rho \in R} \operatorname{supp} \rho\Big). \tag{1.6}$$

When the intersection in (1.6) is empty, or Q or R are not centered, fix $c_{QR} \in \mathbb{C}$ arbitrarily. Define

$$v_{QR} = m \int_0^{c_{QR}} (y_R + \tau)^{\otimes m-1} d\tau \in (L\mathbb{C})^{\otimes m-1}, \tag{1.7}$$

and with the polarization $\mathcal{E}_m$ of E_m from (1.2) consider

$$\mathcal{E}_m\Big(\sum_{R=(\rho_1,\ldots,\rho_m)} (\rho_1 \otimes \ldots \otimes \rho_m)\big(y_R^{\otimes m} + 1 \otimes \sum_{S=(\sigma_2,\ldots,\sigma_m)} (\sigma_2 \otimes \ldots \otimes \sigma_m) v_{SR}\big)\Big); \tag{1.8}$$

we sum over all m–tuples R and $(m-1)$–tuples S in P. (We will not need it, but here is an explanation of (1.8). Say that tensors $w, w' \in L^{(-)}\mathbb{C}^{\otimes m}$ are congruent, $w \equiv w'$, if $w-w'$ is the sum of a degenerate tensor and of monomials $x_1 \otimes \ldots \otimes x_m$ with $\bigcap \operatorname{supp} x_j = \emptyset$. Denote by ∂^m the linear map $(L\mathbb{C})^{\otimes m} \to (L^-\mathbb{C})^{\otimes m}$ defined by $\partial^m(y_1 \otimes \ldots \otimes y_m) = \dot{y}_1 \otimes \ldots \otimes \dot{y}_m$. Then the symmetrization of the argument of $\mathcal{E}_m$ in (1.8) is a solution w of the congruence $\partial^m w \equiv x^{\otimes m}$, in fact it is the unique symmetric solution, up to congruence. It follows that for the $\tilde{E}_m$ sought, $\tilde{E}_m(x)$ must be equal to $\mathcal{E}_m(w)$, which, in turn, equals (1.8).)

We claim that the value in (1.8) depends only on x (and $\mathcal{E}_m$), but not on the partition of unity P and the local integrals y_R. Indeed, suppose first that the local integrals y_R are changed to $\hat{y}_R$, so that the c_{QR} change to $\hat{c}_{QR}$ and v_{QR} to $\hat{v}_{QR}$; but we do not change P. There are $c_R \in \mathbb{C}$ such that for all centered R

$$\hat{y}_R = y_R + c_R \qquad \text{on } \bigcup_{\rho \in R} \operatorname{supp} \rho.$$

Let

$$u_R = m \int_0^{c_R} (y_R + \tau)^{\otimes m-1} d\tau. \tag{1.9}$$

Clearly $\hat{c}_{QR} = c_{QR} + c_Q - c_R$ if $Q \cup R$ is centered. In this case one computes also

$$\begin{aligned} \frac{1}{m}\hat{v}_{QR} &= \int_0^{\hat{c}_{QR}} (\hat{y}_R + \tau)^{\otimes m-1} d\tau \\ &= \int_0^{c_{QR}} (\hat{y}_R - c_R + \tau)^{\otimes m-1} d\tau - \int_0^{c_R} (\hat{y}_R - c_R + \tau)^{\otimes m-1} d\tau \\ &\quad + \int_0^{c_Q} (\hat{y}_R - c_R + c_{QR} + \tau)^{\otimes m-1} d\tau. \end{aligned} \tag{1.10}$$

Because of Lemma 1.2a, in (1.8) only centered R, and such S that $R \cup S$ is centered, will contribute. When $y_R^{\otimes m}$ is changed to $\hat{y}_R^{\otimes m}$, the corresponding contributions change by

$$\begin{aligned} &\sum_R \mathcal{E}_m\Big(\int_0^{c_R} (\rho_1 \otimes \ldots \otimes \rho_m) \frac{d}{d\tau}(y_R + \tau)^{\otimes m} dt \Big) \\ &\qquad = \sum_R \mathcal{E}_m\Big(\int_0^{c_R} (\rho_1 \otimes \ldots \otimes \rho_m)(m \otimes (y_R + \tau)^{\otimes m-1}) d\tau \Big) \\ &\qquad = \sum_R \mathcal{E}_m((\rho_1 \otimes \ldots \otimes \rho_m)(1 \otimes u_R)). \end{aligned}$$

When v_{QR} is changed to $\hat{v}_{QR}$, in view of (1.10), (1.6), and (1.9), the contribution of the terms in the double sum in (1.8) changes by

$$\begin{aligned} \mathcal{E}_m\Big((m\rho_1 \otimes \rho_2\sigma_2 \otimes \ldots \otimes \rho_m\sigma_m)\big(\int_0^{c_S} (y_S + \tau)^{\otimes m-1} d\tau - \int_0^{c_R} (y_R + \tau)^{\otimes m-1} d\tau \big) \Big) \\ = \mathcal{E}_m((\rho_1 \otimes \rho_2\sigma_2 \otimes \ldots \otimes \rho_m\sigma_m)(1 \otimes u_S - 1 \otimes u_R)). \end{aligned}$$

The net change in (1.8) is therefore

$$\mathcal{E}_m\Big(\sum_{R,S}(\rho_1\otimes\rho_2\sigma_2\otimes\ldots\otimes\rho_m\sigma_m)(1\otimes u_S)\Big)=$$
$$\mathcal{E}_m\Big(\sum_{S}(1\otimes\sigma_2\otimes\ldots\otimes\sigma_m)(1\otimes u_S)\Big)=0$$

by Lemma 1.2b, as needed.

Now to pass from P to another partition of unity P', introduce $\Pi=\{\rho\rho'\colon p\in P,\rho'\in P'\}$. One easily shows that P and Π give rise to the same value in (1.8), hence so do P and P'. Therefore (1.8) indeed depends only on x, and we define $\tilde E_m(x)$ by this value. We proceed to check that $\tilde E_m$ has the required properties.

If $x=\dot y$ then all y_R can be chosen y, and (1.8) gives $\tilde E_m(\dot y)=E_m(y)$. Next suppose $x',x''\in L^-\mathbb{C}$ have disjoint support, and $x=x'+x''$. If the supports of all $\rho\in P$ are sufficiently small, then the local integrals $y'_R,\ y''_R$ of x',x'' can be chosen so that for each R one of them is 0. Hence the local integrals $y_R=y'_R+y''_R$ of x will satisfy $y_R^{\otimes m}=y_R'^{\otimes m}+y_R''^{\otimes m}$, whence $\tilde E_m(x)=\tilde E_m(x')+\tilde E_m(x'')$ follows.

To show that $\sum\tilde E_m$ is convergent and represents a holomorphic function, note that $\tilde E_m(x)$ is the sum of terms

$$\mathcal{E}_m(\rho_1 y_R\otimes\ldots\otimes\rho_m y_R)\qquad\text{and}$$
$$\int_0^1\mathcal{E}_m(\rho_1 c_{SR}\otimes\rho_2\sigma_2(y_R+c_{SR}\tau)\otimes\ldots\otimes\rho_m\sigma_m(y_R+c_{SR}\tau))d\tau \tag{1.11}$$

(we have substituted $c_{QR}\tau$ for τ in (1.7)). Since $y_R\in L\mathbb{C}$ and $c_{QR}\in\mathbb{C}$ can be chosen to depend on x in a continuous linear way, each $\tilde E_m$ is a homogeneous polynomial of degree m. Furthermore, let $K\subset L^-\mathbb{C}$ be compact. For each $x\in K,\ m\in\mathbb{N}$, and m–tuples Q,R in P we can choose y_R and c_{QR} so that all the functions

$$\rho c_{QR},\ \rho\rho'(y_R+c_{QR}\tau),$$

$\rho,\rho'\in P,\ 0\le\tau\le1$, belong to some compact $H\subset L\mathbb{C}$. By passing to the balanced hull, it can be assumed H is balanced. If $\lambda>0$, (1.1) implies

$$\max_H|E_m|=\lambda^{-m}\max_{\lambda H}|E_m|\le\lambda^{-m}\max_{\lambda H}|E|=A\lambda^{-m},$$

so that by (1.2)

$$|\mathcal{E}_m(z_1\otimes\ldots\otimes z_m)|\le A(m^m/m!)\ \lambda^{-m}\le A(e/\lambda)^m,$$

if each $z_\mu\in H$. Thus each term in (1.11) satisfies this estimate. If $|P|$ denotes the cardinality of P, we obtain, in view of (1.8)

$$\max_K|\tilde E_m|\le(|P|^m+m|P|^{2m-1})A(e/\lambda)^m.$$

Choosing $|\lambda| > e|P|^2$ we conclude that $\sum \tilde{E}_m$ uniformly converges on K, and, K being arbitrary, $\tilde{E} = \sum \tilde{E}_m$ is holomorphic. By what we have already proved for $\tilde{E}_m$, $\tilde{E} \in \tilde{\mathfrak{E}}$, and $\tilde{E}(\dot{y}) = E(y)$. The above estimates also show that the map $E \to \tilde{E}$ is continuous and linear, which completes the proof of Proposition 1.3.

Now consider an $F \in \mathcal{O}(\mathbb{C} \times L^{(-)}\mathbb{C})$ and its posthomogeneous expansion

$$F = \sum_0^\infty F_m, \quad F_m(\zeta, y) = \int_0^1 F(\zeta, e^{2\pi i\tau} y) e^{-2m\pi i\tau} d\tau. \tag{1.12}$$

Proposition 1.4. *The function F satisfies condition (1) of the Introduction if and only if each F_m is a polynomial in ζ, of degree $\le 2m-2$ (in particular, $F_0 = 0$).*

Proof. As F satisfies (1) precisely when each F_m does, the statement is obvious.

Proof of Theorem 1.1. Apply Proposition 1.3 on each slice $\{\zeta\} \times L^{(-)}\mathbb{C}$. Accordingly, an $\tilde{F}$ in the kernel is posthomogeneous of degree 1, hence, by Proposition 1.4, independent of ζ. Thus indeed $\tilde{F}(\zeta, x) = \text{const} \int x$. Further, the slicewise right inverse applied to $F \in \mathfrak{F}$ clearly produces an additive $\tilde{F} \in \mathcal{O}(\mathbb{C} \times L\mathbb{C})$. To see that $\tilde{F}$ also verifies condition (1) of the Introduction, expand F in a posthomogeneous series

$$F(\zeta, y) = \sum_{m=1}^\infty F_m(\zeta, y) = \sum_{m=1}^\infty \sum_{\nu=0}^{2m-2} \zeta^\nu E_{m\nu}(y), \tag{1.13}$$

by Proposition 1.4, so that

$$\tilde{F}(\zeta, x) = \sum_{m=1}^\infty \sum_{\nu=0}^{2m-2} \zeta^\nu \tilde{E}_{m\nu}(x),$$

with $\tilde{E}_{m\nu}$ m–homogeneous. Again by Proposition 1.4, $\tilde{F}$ verifies condition (1), and so $\in \tilde{\mathfrak{F}}$.

Theorem 1.1 can be used effectively to describe elements of the space $\mathfrak{F}$. With ulterior motives we switch notation $m = n+1$, and consider a homogeneous polynomial $\tilde{E} \in \mathcal{O}(L^-\mathbb{C})$ of degree $n+1 \ge 1$. Its polarization $\mathcal{E}$ defines a distribution D on the torus $(S^1)^{n+1} = T$. Indeed, denote the coordinates on T by $t_j \in \mathbb{R}/\mathbb{Z}$ and set

$$\langle D, \prod_{j=0}^n e^{2\pi i\nu_j t_j} \rangle = \mathcal{E}(x_0 \otimes \ldots \otimes x_n), \qquad x_j(\tau) = e^{2\pi i\nu_j\tau},\ \nu_j \in \mathbb{Z}. \tag{1.14}$$

Since $\tilde{E}$ is continuous,

$$|\mathcal{E}(x_0 \otimes \ldots \otimes x_n)| \le c \prod \|x_j\|_{C^q(S^1)} \qquad \text{with some } c > 0 \text{ and } q \in \mathbb{N}.$$

Hence (1.14) can be estimated, in absolute value, by $c' \prod_j (1 + |\nu_j|)^q$, and it follows by Fourier expansion that D extends to a unique linear form on $C^\infty(T)$. Clearly, D is symmetric, i.e., invariant under permutation of the factors S^1 of T. Also,

$$\mathcal{E}(x_0 \otimes \ldots \otimes x_n) = \langle D, x_0 \otimes \ldots \otimes x_n \rangle, \tag{1.15}$$

if on the right $x_0 \otimes \ldots \otimes x_n$ is identified with the function $\prod x_j(t_j)$.

Assume now $\tilde{E} \in \tilde{\mathfrak{E}}$. Lemma 1.2a implies D is supported on the diagonal of T. The form of distributions supported on submanifolds is in general well understood; in the case at hand, e.g. [H, Theorem 2.3.5] gives that D is a finite sum of distributions of form

$$C^\infty(T) \ni \rho \mapsto \left\langle \Psi, \left. \frac{\partial^{\alpha_1 + \ldots + \alpha_n} \rho}{\partial t_1^{\alpha_1} \ldots \partial t_n^{\alpha_n}} \right|_{\text{diag}} \right\rangle, \quad \alpha_j \geq 0,$$

where Ψ is a distribution on the diagonal of T. In view of Theorem 1.1 and (1.12)–(1.13) we therefore proved

Theorem 1.5. *The restriction of an $(n+1)$–posthomogeneous $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, to $\mathbb{C} \times C^\infty(S^1)$ is a finite sum of functions of form*

$$f(\zeta, y) = \zeta^\nu \left\langle \Phi, \prod_0^n y^{(d_j)} \right\rangle, \qquad \nu \leq 2n, \ d_j \geq d_0 = 1, \ \textit{resp.}\ 0,$$

where Φ is a distribution on S^1. For a general $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, the restriction $F|\mathbb{C} \times C^\infty(S^1)$ is the limit, in the topology of $\mathcal{O}(\mathbb{C} \times C^\infty(S^1))$, of finite sums of the above functions.

2. The G–action on $\mathfrak{F}$

For $g \in G$ let $J_g(\zeta) = dg\zeta/d\zeta$. By considering the posthomogeneous expansion (1.12)–(1.13) of $F \in \mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$, one checks that the function gF defined by

$$(gF)(\zeta, y) = F(g\zeta, \ y/J_g(\zeta)) J_g(\zeta) \tag{2.1}$$

extends to all of $\mathbb{C} \times L^{(-)}\mathbb{C}$, and the extension (also denoted gF) belongs to $\mathfrak{F}$, resp. $\tilde{\mathfrak{F}}$. The action thus defined makes $\mathfrak{F}, \tilde{\mathfrak{F}}$ holomorphic G–modules. The n'th isotypical subspace $\mathfrak{F}^n$, resp. $\tilde{\mathfrak{F}}^n$ is the subspace of $(n+1)$–posthomogeneous functions. In this section we shall describe the space $\mathfrak{F}^0$, and, for $W^{1,p}$ loop spaces, the spaces $\mathfrak{F}^n$ as well, $n \geq 1$.

Theorem 2.1. $\mathfrak{F}^0 \approx (L^-\mathbb{C})^*/\mathbb{C}$, *the dual endowed with the compact–open topology. If $L^-\mathbb{C}$ is interpreted as the space of one–forms on S^1 of the corresponding regularity, then the isomorphism is Diff S^1 equivariant.*

Proof. Indeed, the map $(L^-\mathbb{C})^* = \tilde{\mathfrak{F}}^0 \to \mathfrak{F}^0$ associating with $\Phi \in (L^-\mathbb{C})^*$ the function $F(y) = \langle \Phi, \dot{y} \rangle$ (or $\langle \Phi, dy \rangle$) has one dimensional kernel and a right inverse by Theorem 1.1.

Theorem 2.2. *In the case of $W^{1,p}$ loop spaces $\mathfrak{F} = \bigoplus_{n \le p-1} \mathfrak{F}^n$. Furthermore*

$$\mathfrak{K}_n \otimes L^{p/(n+1)}(S^1)^* \approx \mathfrak{F}^n, \qquad 1 \le n \le p-1,$$

as G–modules, G acting on $L^{p/(n+1)}(S^1)^*$ trivially. Indeed, the map $\varphi \otimes \Phi \mapsto F$ given by

$$F(\zeta, y) = \psi(\zeta)\langle \Phi, \dot{y}^{n+1}\rangle, \qquad \varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n}, \tag{2.2}$$

induces the isomorphism above. (To achieve Diff S^1 equivariant isomorphism, replace $L^{p/(n+1)}(S^1)$ by the space $L^{p/(n+1)}_{n+1}(S^1)$ of $(n+1)$–differentials.)

We shall need a few auxiliary results to prove the theorem.

Lemma 2.3. *Let $m \ge 2$ be an integer and Ψ a distribution on S^1. If the function*

$$C^\infty(S^1) \ni x \mapsto \langle \Psi, x^m \rangle \in \mathbb{C} \tag{2.3}$$

extends to a homogeneous polynomial E on $L^p(S^1)$ then $\Psi \equiv 0$, or $m \le p$ and Ψ extends to a form Φ on $L^{p/m}(S^1)$. In the latter case the map $E \mapsto \Phi$ is continuous linear.

Proof. There is a constant C such that

$$|\langle \Psi, x^m \rangle| = |E(x)| \le C\Big(\int |x|^p\Big)^{m/p}, \qquad x \in C^\infty(S^1). \tag{2.4}$$

Let $z \in C^\infty(S^1)$ be real valued and $x_\epsilon = (z + i\epsilon)^{1/m} \in C^\infty(S^1)$. By (2.4)

$$|\langle \Psi, z \rangle| = \lim_{\epsilon \to 0} |\langle \Psi, x_\epsilon^m \rangle| \le C\Big(\int |z|^{p/m}\Big)^{m/p}.$$

As the same estimate holds for imaginary z, it will hold for a general $z \in C^\infty(S^1)$ too, perhaps with a different C. Therefore Ψ extends to a form Φ on $L^{p/m}(S^1)$. Unless $p \ge m$, $\Phi = 0$ by Day's theorem [D]. With $z \in L^{p/m}(S^1)$, any choice of measurable m'th root $z^{1/m}$, and $y_\varepsilon \in C^\infty(S^1)$ converging to $z^{1/m}$ in L^p,

$$\langle \Phi, z \rangle = \lim_{\varepsilon \to 0} \langle \Phi, y_\varepsilon^m \rangle = \lim_{\varepsilon \to 0} E(y_\varepsilon) = E(z^{1/m}).$$

This shows that Φ is uniquely determined by E, and depends continuously and linearly on E.

In the rest of this section we work with $W^{1,p}$ loop spaces. Write $\mathfrak{E}^n \subset \mathfrak{E}$, $\tilde{\mathfrak{E}}^n \subset \tilde{\mathfrak{E}}$ for the space of $(n+1)$–homogeneous functions.

Lemma 2.4. *If $m \geq 2$ and $E \in \tilde{\mathfrak{E}}^{m-1} \subset \mathcal{O}(L^p(S^1))$, then $E(x) = \langle \Phi, x^m \rangle$ with a unique $\Phi \in L^{p/m}(S^1)^*$. In particular, $E = 0$ if $m > p$. Also, the map $E \mapsto \Phi$ is an isomorphism between $\tilde{\mathfrak{E}}^{m-1}$ and $L^{p/m}(S^1)^*$.*

Proof. We shall prove by induction, first assuming $m = 2$. By Theorem 1.5 there are distributions Φ_α so that

$$E(x) = \sum_{\alpha=0}^{d} \langle \Phi_\alpha, x x^{(\alpha)} \rangle, \quad x \in C^\infty(S^1).$$

Now any $x^{(\alpha)} x^{(\beta)}$ will be a linear combination of expressions $(x^{(j)} x^{(j)})^{(h)}$, as one easily proves by induction of $|\alpha - \beta|$. It follows that E can be written with distributions Ψ_j as

$$E(x) = \sum_{j=0}^{d} \langle \Psi_j, (x^{(j)})^2 \rangle, \quad x \in C^\infty(S^1). \tag{2.5}$$

Next we show that $d = 0$.

Indeed, assuming $d > 0$, for fixed $x \in C^\infty(S^1)$

$$E(\cos \lambda x) + E(\sin \lambda x) = \lambda^{2d} \langle \Psi_d, \dot{x}^{2d} \rangle + \sum_{j=0}^{2d-1} c_j(x) \lambda^d \tag{2.6}$$

is a polynomial in λ. For fixed $\lambda \in \mathbb{C}$ the maps $x \mapsto \cos \lambda x$, $x \mapsto \sin \lambda x$ map the Banach algebra $W^{1,1}(S^1)$ holomorphically into itself, hence into $L^p(S^1)$. Therefore the left hand side of (2.6) extends to $W^{1,1}(S^1)$, and $\langle \Psi_d, \dot{x}^{2d} \rangle$ must also. The extension of this latter will be an additive, $2d$–homogeneous polynomial E' on $W^{1,1}(S^1)$, satisfying $E'(x + \text{const}) = E'(x)$. By Proposition 1.3 there is therefore a unique additive $2d$–homogeneous polynomial $\tilde{E}$ on $W^{0,1}(S^1) = L^1(S^1)$ such that $E'(x) = \tilde{E}(\dot{x})$. Since the restriction $\tilde{E}|C^\infty(S^1)$ is also unique,

$$\tilde{E}(x) = \langle \Psi_d, x^{2d} \rangle, \qquad x \in C^\infty(S^1).$$

In particular, the expression on the right continuously extends to $L^1(S^1)$. By virtue of Lemma 2.3, $\Psi_d \equiv 0$. Thus (2.5) reduces to $E(x) = \langle \Psi, x^2 \rangle$, $x \in C^\infty(S^1)$, and by another application of Lemma 2.3, Ψ extends to a form Φ on $L^{p/2}(S^1)$.

Now assume the Lemma is known for degree $m - 1 \geq 2$, and consider an $E \in \tilde{\mathfrak{E}}^{m-1}$ and its polarization $\mathcal{E}$. For fixed $x_1 \in C^\infty(S^1)$ the inductive assumption implies that there is a distribution Θ such that $\mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \langle \Theta, \prod_2^m x_j \rangle$; in particular,

$$\mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \mathcal{E}(x_1 \otimes \prod_2^m x_j \otimes 1 \otimes \ldots \otimes 1), \qquad x \in C^\infty(S^1).$$

The case $m = 2$ now gives a distribution Ψ such that $\mathcal{E}(x_1 \otimes \ldots \otimes x_m) = \langle \Psi, \prod_1^m x_j \rangle$. We conclude by Lemma 2.3: Ψ extends to $\Phi \in L^{p/m}(S^1)^*$, and $\Phi = 0$ unless $m \leq p$. It is

clear that Φ is uniquely determined by E, and the map $\tilde{\mathfrak{E}}^{m-1} \ni E \mapsto \Phi \in L^{p/m}(S^1)^*$ is an isomorphism.

Proof of Theorem 2.2. To construct the inverse of the map defined by (2.2), write an arbitrary $F \in \mathfrak{F}^n,\ n \geq 1$, as

$$F(\zeta, y) = \sum_{\nu=0}^{2n} \zeta^\nu E_\nu(y), \quad E_\nu \in \mathfrak{E}^n,$$

cf. Proposition 1.4, and find the unique $\tilde{E}_\nu \in \tilde{\mathfrak{E}}^n$ so that $E_\nu(y) = \tilde{E}_\nu(\dot{y})$, see Proposition 1.3. By Lemma 2.4 there are unique $\Phi_\nu \in L^{p/(n+1)}(S^1)^*$ such that $\tilde{E}_\nu(x) = \langle \Phi_\nu, x^{n+1} \rangle$. If $p < n+1$ then $\Phi_\nu = 0$ and so $\mathfrak{F}^n = (0)$. Otherwise the map

$$\mathfrak{F}^n \ni F \mapsto \sum_0^{2n} \zeta^\nu (d\zeta)^{-n} \otimes \Phi_\nu \in \mathfrak{K}_n \otimes L^{p/(n+1)}(S^1)^*$$

is the inverse of the map given in (2.2), so (2.2) indeed induces an isomorphism. Finally, the posthomogeneous expansion of an arbitrary $F \in \mathfrak{F}$ is

$$F = \sum_0^\infty F_n = \sum_0^{[p-1]} F_n,$$

which completes the proof.

3. Cuspidal cocycles

In this section we shall construct an isomorphism between $H^{0,1}(L\mathbb{P}_1)$ and a space of holomorphic Čech cocycles on $L\mathbb{P}_1$. We represent $\mathbb{P}_1$ as $\mathbb{C} \cup \{\infty\}$. Constant loops constitute a submanifold of $L\mathbb{P}_1$, that we identify with $\mathbb{P}_1$. If $a, b, \ldots \in \mathbb{P}_1$, set $U_{ab\ldots} = \mathbb{P}_1 \backslash \{a, b, \ldots\}$. Thus $LU_a,\ a \in \mathbb{P}_1$, form an open cover of $L\mathbb{P}_1$, with $LU_\infty = L\mathbb{C}$ a Fréchet algebra. If $g \in G$ then $g(LU_a) = LU_{ga}$.

Suppose we are given $v\colon \mathbb{P}_1 \to \mathbb{C}$, finitely many $a, b, \ldots \in \mathbb{P}_1$, and a function $u : LU_{ab\ldots} \to \mathbb{C}$. If ∞ is among $a, b, \ldots$, let us say that u is v-cuspidal at ∞ if $u(x + \lambda) \to v(\infty)$ as $\mathbb{C} \ni \lambda \to \infty$, for all $x \in LU_{ab\ldots}$; and in general, that u is v-cuspidal if g^*u is g^*v-cuspidal at ∞ for all $g \in G$ that maps one of $a, b, \ldots$ to ∞. When $v \equiv 0$ we simply speak of cuspidal functions.

Proposition 3.1. *Given a closed $f \in C^\infty_{0,1}(L\mathbb{P}_1)$ and $v \in C^\infty(\mathbb{P}_1)$ such that $\overline{\partial} v = f|\mathbb{P}_1$, for each $a \in \mathbb{P}_1$ there is a unique v-cuspidal $u_a \in C^\infty(LU_a)$ that solves $\overline{\partial} u_a = f|LU_a$. Furthermore, $u_a|U_a = v|U_a$, and $u(a, x) = u_a(x)$ is smooth in (a, x), holomorphic in a.*

Proof. Uniqueness follows since for fixed $g \in G,\ y \in L\mathbb{C}$, on the line $\{g(y+\lambda)\colon \lambda \in \mathbb{P}_1\}$ the $\overline{\partial}$ equation is uniquely solvable up to an additive constant, which constant is determined

by the cuspidal condition. To construct u_a, fix a $g \in G$ with $g\infty = a$, let $Y = \{y \in L\mathbb{C} : y(0) = 0\}$ and

$$P_g\colon \mathbb{P}_1 \times Y \ni (\lambda, y) \mapsto g(y+\lambda) \in L\mathbb{P}_1,$$

a biholomorphism between $\mathbb{C} \times Y$ and LU_a. Setting $f_g = P_g^* f$, by [L1, Theorem 5.4] on the $\mathbb{P}_1$ bundle $\mathbb{P}_1 \times Y$ the equation $\overline{\partial} u_g = f_g$ has a unique smooth solution satisfying $u_g(\infty, x) = v(a)$. It follows that $u_a = (P_g^{-1})^*(u_g|\mathbb{C} \times Y)$ solves $\overline{\partial} u_a = f|LU_a$. Also, $g^* u_a$ is g^*v–cuspidal at ∞. On U_a both u_a and v solve the same $\overline{\partial}$–equation, and have the same limit at a, hence $u_a|U_a = v|U_a$.

One can also consider

$$P\colon \mathbb{P}_1 \times G \times Y \ni (\lambda, g, y) \mapsto g(y+\lambda) \in L\mathbb{P}_1$$

and $f' = P^* f$. Again by [L1, Theorem 5.4], on the $\mathbb{P}_1$ bundle $\mathbb{P}_1 \times G \times Y$ the equation $\overline{\partial} u' = f'$ has a smooth solution satisfying $u'(\infty, g, x) = v(g\infty)$. Uniqueness of u_g implies $u'(\lambda, g, x) = u_g(\lambda, x)$, whence $u_g(\lambda, x)$ depends smoothly on (λ, g, x), and $u_a(x)$ on (a, x). Furthermore, u' is holomorphic on $P^{-1}(x)$ for any x. In particular, if $g \in G$ with $g\infty = a$ is chosen to depend holomorphically on a (which can be done locally), then it follows that $u_a(x) = u'(g^{-1}x(0), g, g^{-1}x - g^{-1}x(0))$ is holomorphic in a.

Since f determines v up to an additive constant, we can uniquely associate with f the Čech cocycle $\mathfrak{f} = (u_a - u_b : a, b \in \mathbb{P}_1)$. The components of $\mathfrak{f}$ are cuspidal holomorphic functions on LU_{ab}. One easily verifies

Proposition 3.2. *f is exact if and only if $\mathfrak{f} = 0$. Hence $\mathfrak{f}$ depends only on the cohomology class $[f] \in H^{0,1}(L\mathbb{P}_1)$. The components $h_{ab}([f], x)$ of $\mathfrak{f}$ depend holomorphically on $a, b \in \mathbb{P}_1$ and $x \in LU_{ab}$, and satisfy the transformation formula*

$$h_{ga,gb}([f], gx) = h_{ab}(g^*[f], x), \qquad g \in G,\ x \in LU_{ab}. \tag{3.1}$$

Set

$$\Omega = \{(a, b, x) \in \mathbb{P}_1 \times \mathbb{P}_1 \times L\mathbb{P}_1 : a, b \notin x(S^1)\}.$$

Let $\mathfrak{H}$ denote the space of those holomorphic cocycles $\mathfrak{h} = (\mathfrak{h}_{ab})_{a,b \in \mathbb{P}_1}$ of the covering $\{LU_a\}$, for which $\mathfrak{h}_{ab}(x)$ depends holomorphically on a, b, and $x \in LU_{ab}$, and each $\mathfrak{h}_{ab}$ is cuspidal. Then $\mathfrak{H} \subset \mathcal{O}(\Omega)$, with the compact open topology, is a complete, separated, locally convex space. The action of G on Ω induces a G–module structure on $\mathfrak{H}$:

$$(g^*\mathfrak{h})_{ab}(x) = \mathfrak{h}_{ga,gb}(gx), \qquad g \in G. \tag{3.2}$$

Proposition 3.2 implies the map $[f] \mapsto \mathfrak{f}$ is a monomorphism $H^{0,1}(L\mathbb{P}_1) \to \mathfrak{H}$ of G–modules.

Theorem 3.3. *The map $[f] \mapsto \mathfrak{f}$ is an isomorphism $H^{0,1}(L\mathbb{P}_1) \to \mathfrak{H}$.*

The proof would be routine if the loop space $L\mathbb{P}_1$ admitted smooth partitions of unity; but a typical loop space does not, see [K]. The proof that we offer here will work only

when the loops in $L\mathbb{P}_1$ are of regularity $W^{1,3}$ at least, and we shall return to the case of $L_{1,p}(\mathbb{P}_1)$, $p<3$, in Section 6.

Those $g\in G$ that preserve the Fubini–Study metric form a subgroup (isomorphic to) SO(3). Denote the Haar probability measure on SO(3) by dg.

Lemma 3.4. *Unless $L\mathbb{P}_1 = L_{1,p}\mathbb{P}_1,\ p<3$, there is a $\chi\in C^\infty(L\mathbb{P}_1)$ such that $\chi=0$ in a neighborhood of $L\mathbb{P}_1\setminus L\mathbb{C}=\{x\colon \infty\in x(S^1)\}$, and $\int_{\mathrm{SO}(3)} g^*\chi dg=1$.*

Proof. With $c_0\in(0,\infty)$ to be specified later, fix a nonnegative $\rho\in C^\infty(\mathbb{R})$ such that $\rho(\tau)=1$, resp. 0 when $|\tau|<c_0$, resp. $>2c_0$. For $x\in L\mathbb{C}$ let

$$\psi(x)=\rho\left(\int_{S^1}(1+|x|^2)^{3/4}\right),$$

and define $\psi(x)=0$ if $x\in L\mathbb{P}_1\setminus L\mathbb{C}$. We claim that ψ vanishes in a neighborhood of an arbitrary $x\in L\mathbb{P}_1\setminus L\mathbb{C}$. This will then also imply that $\psi\in C^\infty(L\mathbb{P}_1)$.

Indeed, suppose $x(t_0)=\infty$. In a neighborhood of $t_0\in S^1$ the function $z=1/x$ is $W^{1,3}$, hence Hölder continuous with exponent $2/3$ by the Sobolev Embedding Theorem, [H, Theorem 4.5.12]. In this neighborhood therefore $|x(t)|\ge c|t-t_0|^{-2/3}$, and $\int_{S^1}(1+|x|^2)^{3/4}=\infty$. When $y\in L\mathbb{C}$ is close to x, $\int_{S^1}(1+|y|^2)^{3/4}>2c_0$, i.e. $\psi(y)=0$.

Next we show that for every $x\in L\mathbb{P}_1$ there is a $g\in\mathrm{SO}(3)$ with $\psi(gx)>0$. Let $d(a,b)$ denote the Fubini–Study distance between $a,b\in\mathbb{P}_1$; then with some $c>0$

$$1+|\zeta|^2\le\frac{c}{d(\zeta,\infty)^2},\qquad\text{and}\qquad\int_{S^1}(1+|x|^2)^{3/4}\le c\int_{S^1}d(x,\infty)^{-3/2}.$$

Hence

$$\int_{\mathrm{SO}(3)}\int_{S^1}(1+|gx(t)|^2)^{3/4}dtdg\le c\int_{S^1}\int_{\mathrm{SO}(3)}d(gx(t),\infty)^{-3/2}dgdt=cI,$$

where, for any $\zeta\in\mathbb{P}_1$

$$I=\int_{\mathrm{SO}(3)}d(g\zeta,\infty)^{-3/2}dg=\int_{\mathbb{P}_1}d(\cdot,\infty)^{-3/2}<\infty,$$

the last integral with respect to the Fubini–Study area form. If c_0 is chosen $>cI$ then indeed $\int_{S^1}(1+|gx|^2)^{3/4}<c_0$ and $\psi(gx)=1$ for some $g\in\mathrm{SO}(3)$.

It follows that $\int_{\mathrm{SO}(3)}\psi(gx)dg>0$, and we can take $\chi(x)=\psi(x)/\int_{\mathrm{SO}(3)}\psi(gx)dg$.

Proof of Theorem 3.3. Given $\mathfrak{h}\in\mathfrak{H}$, extend the functions $(g^*\chi)\mathfrak{h}_{a,g\infty}$ from $LU_{a,g\infty}$ to LU_a by zero, and define the cuspidal functions

$$u_a=\int_{\mathrm{SO}(3)}(g^*\chi)\mathfrak{h}_{a,g\infty}dg,\quad a\in\mathbb{P}_1.$$

Then $u_a-u_b=\int_{\mathrm{SO}(3)}(g^*\chi)\mathfrak{h}_{ab}dg=\mathfrak{h}_{ab}$, so that $f=\overline{\partial}u_a$ on LU_a consistently defines a closed $f\in C^\infty_{0,1}(L\mathbb{P}_1)$. It is immediate that the map $\mathfrak{h}\mapsto[f]\in H^{0,1}(L\mathbb{P}_1)$ is left inverse to the monomorphism $[f]\mapsto\mathfrak{f}$, whence the theorem follows.

4. The map $\mathfrak{H} \to \mathfrak{F}$

Consider an $\mathfrak{h} = (\mathfrak{h}_{ab}) \in \mathfrak{H}$. The cocycle relation implies that $d_\zeta \mathfrak{h}_{a\zeta}(x)$ is independent of a; for $\zeta \in \mathbb{C}$ we can write it as

$$d_\zeta \mathfrak{h}_{a\zeta}(x) = F\Big(\zeta, \frac{1}{\zeta - x}\Big) d\zeta, \qquad x \in LU_\zeta, \tag{4.1}$$

where $F \in \mathcal{O}(\mathbb{C} \times L\mathbb{C})$. Set $F = \alpha(\mathfrak{h})$. Since $\mathfrak{h}_{aa} = 0$,

$$\mathfrak{h}_{ab}(x) = \int_a^b F\Big(\zeta, \frac{1}{\zeta - x}\Big) d\zeta, \tag{4.2}$$

provided a, b are in the same component of $\mathbb{P}_1 \backslash x(S^1)$—that we shall express by saying x does not separate a, b—, and we integrate along a path within this component. The main result of this section is

Theorem 4.1. $\alpha(\mathfrak{h}) = F \in \mathfrak{F}$.

The heart of the matter will be the special case when $\mathfrak{h}$ is in an irreducible submodule $\approx \mathfrak{K}_n$. A vector that corresponds in this isomorphism to $\mathrm{const}(d\zeta)^{-n} \in \mathfrak{K}_n$ is said to be of lowest weight $-n$. Thus, if $\mathfrak{l}$ is of lowest weight $-n \le 0$, then

$$g_\lambda^* \mathfrak{l} = \lambda^{-n} \mathfrak{l}, \quad \text{when } g_\lambda \zeta = \lambda\zeta, \quad \lambda \in \mathbb{C}, \text{ and} \tag{4.3}$$

$$g_\lambda^* \mathfrak{l} = \mathfrak{l}, \qquad \text{when } g_\lambda \zeta = \zeta + \lambda, \quad \lambda \in \mathbb{C}. \tag{4.4}$$

Conversely, an $\mathfrak{l} \neq 0$ satisfying (4.3), (4.4) is a lowest weight vector and spans an irreducible submodule, isomorphic to $\mathfrak{K}_n$, but we shall not need this fact.

If $\mathfrak{l} \in \mathfrak{H}$ satisfies (4.4) then $\mathfrak{l}_{\infty\zeta}(x) = \mathfrak{l}_{\infty,\zeta+\lambda}(x + \lambda)$ by (3.2), whence $d_\zeta \mathfrak{l}_{\infty\zeta}(x)$ depends only on $\zeta - x$, and $\alpha(\mathfrak{l})$ is of form $F(\zeta, y) = E(y)$. If, in addition, $\mathfrak{l}$ satisfies (4.3), then similarly it follows that $E \in \mathcal{O}(L\mathbb{C})$ is homogeneous of degree $n+1$. We now fix a nonzero lowest weight vector $\mathfrak{l} \in \mathfrak{H}$, the corresponding $(n+1)$–homogeneous polynomial E, and its polarization $\mathcal{E}$, cf. (1.2).

Proposition 4.2. $\mathcal{E}(1 \otimes y_1 \otimes \ldots \otimes y_n) = 0$, *and so* $E(y + \mathit{const}) = E(y)$.

Proof. Since $\mathfrak{l}_{\infty 0} \in \mathcal{O}(LU_{\infty 0})$ is cuspidal and homogeneous of order $-n$,

$$0 = \lim_{\lambda \to \infty} \mathfrak{l}_{\infty 0}\Big(\frac{1}{\lambda + x}\Big) = \lim_{\lambda \to \infty} \lambda^n \mathfrak{l}_{\infty 0}\Big(\frac{1}{1 + x/\lambda}\Big).$$

Thus $\mathfrak{l}_{\infty 0}$ vanishes at 1 to order $\ge n+1$. Hence

$$\frac{\partial}{\partial \zeta}\Big|_{\zeta=0} \mathfrak{l}_{\infty 0}(x - \zeta) = \frac{\partial}{\partial \zeta}\Big|_{\zeta=0} \mathfrak{l}_{\infty\zeta}(x) = E\Big(\frac{1}{x}\Big)$$

vanishes at $x = 1$ to order $\ge n$, and the same holds for $E(x)$. Differentiating E in the directions $y_1, \ldots, y_n$, we obtain at $x = 1$, as needed, that $n! \mathcal{E}(1 \otimes y_1 \otimes \ldots \otimes y_n) = 0$.

Let $\mathfrak{K}_n \ni \varphi \mapsto \mathfrak{h}^\varphi \in \mathfrak{H}$ denote the homomorphism that maps $(d\zeta)^{-n}$ to $\mathfrak{l}$.

Proposition 4.3.

$$d_\zeta \mathfrak{h}^\varphi_{a\zeta}(x) = \psi(\zeta) E\Big(\frac{1}{\zeta - x}\Big) d\zeta, \qquad \varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n}. \tag{4.5}$$

By homogeneity, the right hand side can also be written $\varphi(\zeta)E(d\zeta/(\zeta - x))$.

Proof. Denote the form on the left hand side of (4.5) by ω^φ. In view of (3.2) it transforms under the action of G on $\mathbb{P}_1 \times L\mathbb{P}_1$ as

$$g^*\omega^\varphi = \omega^{g\varphi}, \qquad g \in G. \tag{4.6}$$

If we show that the right hand side of (4.5) transforms the same way, then (4.5) will follow, since it holds when $\psi \equiv 1$, see (4.1). In fact, it will suffice to check the transformation formula for $g\zeta = \lambda\zeta,\ g\zeta = \zeta + \lambda$ ($\lambda \in \mathbb{C}$), and $g\zeta = 1/\zeta$, maps that generate G. We shall do this for the last map, the most challenging of the three types. The pullback of the right hand side of (4.5) by $g\zeta = 1/\zeta$ is

$$\begin{aligned}(g\varphi)(\zeta)E\Big(\frac{d(g\zeta)}{g\zeta - gx}\Big) &= (g\varphi)(\zeta)E\Big(\frac{-d\zeta/\zeta^2}{(1/\zeta) - (1/x)}\Big)\\ &= (g\varphi)(\zeta)E\Big(\frac{d\zeta}{\zeta - x} - \frac{d\zeta}{\zeta}\Big) = (g\varphi)(\zeta)E\Big(\frac{d\zeta}{\zeta - x}\Big),\end{aligned}$$

by Proposition 4.2, which is what we needed.

The form $\mathcal{E}$ defines a symmetric distribution D on the torus $T = (S^1)^{n+1}$ as in Section 1, cf. (1.14). By (1.15), (4.2), and Proposition 4.3

$$\mathfrak{h}^\varphi_{ab}(x) = \int_a^b \psi(\zeta)\Big\langle D,\ \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x}\Big\rangle d\zeta, \qquad \varphi = \psi(\zeta)(d\zeta)^{-n}, \tag{4.7}$$

provided $x \in L_\infty U_{ab}$ does not separate a, b. To prove Theorem 4.1, we have to understand supp D. Let

$$O = \{x \in C^\infty(S^1) \colon \pm i \notin x(S^1)\}, \text{ and } O' = \{x \in O \colon [-i, i] \cap x(S^1) = \emptyset\},$$

where $[-i, i]$ stands for the segment joining $\pm i$.

Lemma 4.4. *With Δ a symmetric distribution on $T = (S^1)^{n+1}$ and $\nu = 0, \ldots, 2n-2$, let*

$$I_\nu(x) = \int_{[-i,i]} \Big\langle \Delta,\ \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x}\Big\rangle \zeta^\nu d\zeta, \qquad x \in O'.$$

If each I_ν continues analytically to O then Δ is supported on the diagonal of T.

In preparation to the proof, consider a holomorphic vector field V on O, and observe that VI_ν also continues analytically to O. Such vector fields can be thought of as holomorphic maps $V\colon O \to C^\infty(S^1)$. Using the symmetry of Δ we compute

$$(VI_\nu)(x) = (n+1)\int_{[-i,i]} \Big\langle \Delta,\ \frac{V(x)}{(\zeta - x)^2} \otimes \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x}\Big\rangle \zeta^\nu d\zeta, \quad x \in O'. \tag{4.8}$$

Proof of Lemma 4.4, case $n = 1$. Let $\overline{s}_0 \neq \overline{s}_1 \in S^1$. To show Δ vanishes near $\overline{s} = (\overline{s}_0, \overline{s}_1)$, construct a smooth family $x_{\varepsilon,s} \in O$ of loops, where $\varepsilon \in [0,1]$ and $s \in T$ is in a neighborhood of $\overline{s}$, so that

$$x_{\varepsilon,s}(\tau) = (-1)^j(\varepsilon^2 + (\tau - s_j)^2), \qquad \text{when } \tau \in S^1 \text{ is near } \overline{s}_j,\ j = 0, 1; \tag{4.9}$$

here, perhaps abusively, $\tau - s_j$ denotes both a point in $S^1 = \mathbb{R}/\mathbb{Z}$ and its representative in $\mathbb{R}$ that is closest to 0. Make sure that $x_{\varepsilon,s} \in O'$ when $\varepsilon > 0$. Fix $y_0, y_1 \in C^\infty(S^1)$ so that $y_j \equiv 1$ near $\overline{s}_j$, and (4.9) holds when τ, s_j are in a neighborhood of supp y_j. This forces y_0, y_1 to have disjoint support. With constant vector fields $V_j = y_j$

$$(V_1 V_0 I_0)(x) = 2\int_{[-i,i]} \Big\langle \Delta,\ \frac{y_0}{(\zeta - x)^2} \otimes \frac{y_1}{(\zeta - x)^2}\Big\rangle d\zeta, \qquad x \in O', \tag{4.10}$$

analytically continues to O. In particular, for $\varepsilon > 0$ and $t = (t_0, t_1) \in T$ setting

$$K_\varepsilon(t,s) = \int_{[-i,i]} \frac{y_0(t_0) y_1(t_1) d\zeta}{(\zeta - x_{\varepsilon,s}(t_0))^2 (\zeta - x_{\varepsilon,s}(t_1))^2}, \qquad s \text{ near } \overline{s},$$

it follows that $\langle \Delta, K_\varepsilon(\cdot, s)\rangle$ stays bounded as $\varepsilon \to 0$. Therefore, if $\rho \in C^\infty(T)$ is supported in a sufficiently small neighborhood of $\overline{s}$,

$$\langle \Delta,\ \varepsilon^4 \int_T K_\varepsilon(\cdot, s)\rho(s) ds\rangle \to 0, \qquad \varepsilon \to 0. \tag{4.11}$$

On the other hand we shall show that for such ρ

$$\varepsilon^4 \int_T K_\varepsilon(\cdot, s)\rho(s) ds \to c\rho, \qquad \varepsilon \to 0, \tag{4.12}$$

in the topology of $C^\infty(T)$; here $c \neq 0$ is a constant.

It will suffice to verify (4.12) on supp $y_0 \otimes y_1$, since both sides vanish on the complement. Thus we shall work on small neighborhoods of $\overline{s}$; we can pretend $\overline{s} \in \mathbb{R}^2$, and work on $\mathbb{R}^2$ instead of T. When $s,\ t \in \mathbb{R}^2$ are close to $\overline{s}$, the left hand side of (4.12) becomes

$$\varepsilon^4 y_0(t_0) y_1(t_1) \int_{\mathbb{R}^2} \int_{[-i,i]} \frac{\rho(s) d\zeta ds}{(\zeta - \varepsilon^2 - (s_0 - t_0)^2)^2 (\zeta + \varepsilon^2 + (s_1 - t_1)^2)^2}. \tag{4.13}$$

Substituting $s = t + \varepsilon u$ and $\zeta = \varepsilon^2 \xi$, we compute the limit in (4.12) is

$$
\begin{aligned}
(4.14) \qquad & \lim_{\varepsilon \to 0} y_0(t_0) y_1(t_1) \int_{\mathbb{R}^2} \int_{[-i/\varepsilon^2, i/\varepsilon^2]} \frac{\rho(t+\varepsilon u) d\xi du}{(\xi - 1 - u_0^2)^2 (\xi + 1 + u_1^2)^2} \\
& = 4\pi i y_0(t_0) y_1(t_1) \int_{\mathbb{R}^2} \frac{\rho(t) du}{(2 + u_0^2 + u_1^2)^3} = c\rho(t),
\end{aligned}
$$

if $y_0 \otimes y_1 = 1$ on $\operatorname{supp} \rho$. This limit is first seen to hold uniformly. However, since the integral operator in (4.13) is a convolution, in (4.14) in fact all derivatives converge uniformly. Now (4.11) and (4.12) imply $\langle \Delta, \rho \rangle = 0$, so that Δ vanishes close to $\overline{s}$, q.e.d.

Proof of Lemma 4.4, general n. The base case $n = 1$ settled and the statement being vacuous when $n = 0$, we prove by induction. Assume the Lemma holds on the n–dimensional torus, and with $y \in C^\infty(S^1)$, consider holomorphic vector fields $V_\mu(x) = yx^\mu$, $\mu = 0, 1, 2$. (These vector fields continue to all of $L\mathbb{P}_1$, and generate the Lie algebra of the loop group LG.) In view of (4.8), for $x \in O'$

$$
(4.15) \qquad \int_{[-i,i]} \Big\langle \Delta, y \otimes \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \Big\rangle \zeta^\nu d\zeta = \frac{1}{n+1} (V_0 I_{\nu+2} - 2V_1 I_{\nu+1} + V_2 I_\nu).
$$

Therefore the left hand side continues analytically to O, provided $\nu = 0, \ldots, 2n-4$. If Δ^y denotes the distribution on $(S^1)^n$ defined by $\langle \Delta^y, \rho \rangle = \langle \Delta, y \otimes \rho \rangle$, the left hand side of (4.15) is

$$
\int_{[-i,i]} \Big\langle \Delta^y, \frac{1}{\zeta - x} \otimes \ldots \otimes \frac{1}{\zeta - x} \Big\rangle \zeta^\nu d\zeta.
$$

The inductive hypothesis implies Δ^y is supported on the diagonal of $(S^1)^n$. This being true for all y, the symmetric distribution Δ itself must be supported on the diagonal.

Corollary 4.5. *The distribution D in (4.7) is supported on the diagonal of T.*

Proof of Theorem 4.1. First assume that $\mathfrak{h} \in \mathfrak{H}$ is in an irreducible submodule $\approx \mathfrak{K}_n$, and $\mathfrak{l} \neq 0$ is a lowest weight vector in this submodule. Thus $\mathfrak{h} = \mathfrak{h}^\varphi$ with some $\varphi \in \mathfrak{K}_n$, $\varphi(\zeta) = \psi(\zeta)(d\zeta)^{-n}$. With $\mathfrak{l}$ we associated an $(n+1)$–homogeneous polynomial E on $L\mathbb{C}$ and a distribution D on $(S^1)^{n+1}$. By Proposition 4.3 $F(\zeta, y) = \psi(\zeta) E(y)$, and so $F(\zeta, y + \text{const}) = F(\zeta, y)$ by Proposition 4.2. Since $\deg \psi \leq 2n$, $F(\zeta/\lambda, \lambda^2 y) = O(\lambda^2)$ as $\lambda \to 0$. Finally, take $x, y \in L\mathbb{C}$ with disjoint support. If $x, y \in C^\infty(S^1)$,

$$
E(x+y) = \langle D, (x+y)^{\otimes n+1} \rangle = \langle D, x^{\otimes n+1} \rangle + \langle D, y^{\otimes n+1} \rangle = E(x) + E(y),
$$

as $\operatorname{supp} D$ is on the diagonal. By approximation $E(x+y) = E(x) + E(y)$ follows in general, whence F itself is additive. We conclude $F \in \mathfrak{F}$ if $\mathfrak{h}$ is in an irreducible submodule.

By linearity it follows that $F \in \mathfrak{F}$ whenever $\mathfrak{h}$ is in the span of irreducible submodules. Since this span is dense in $\mathfrak{H}$ (cf. [BD, III.5.7] and the explanation in the Introduction connecting representations of G with those of the compact group SO(3)), $\alpha(\mathfrak{h}) \in \mathfrak{F}$ for all $\mathfrak{h} \in \mathfrak{H}$.

Theorem 4.6. *The map α is a G–morphism.*

Proof. It suffices to verify that the restriction of α to an irreducible submodule of $\mathfrak{H}$ is a G–morphism, which follows directly from Proposition 4.3.

5. The structure of $\mathfrak{H}$

The main result of this Section is

Theorem 5.1. *The G–morphism $\alpha\colon\mathfrak{H} \to \mathfrak{F}$ has a right inverse β. Its kernel is one dimensional, spanned by the G–invariant cocycle*

$$\mathfrak{h}_{ab}(x) = ind_{ab}x$$

(= the winding number of $x\colon S^1 \to U_{ab}$).

We shall need the following

Lemma 5.2. *With notation as in Section 1, suppose $z_1, \ldots, z_N \in L^-\mathbb{C}$ are such that no point in S^1 is contained in the support of more than two z_j. If $\tilde{F} \in \tilde{\mathfrak{F}}$ then*

$$\tilde{F}(\zeta, \sum_{j=1}^{N} z_j) = \sum_{i<j} \tilde{F}(\zeta, z_i + z_j) - (N-2)\sum_{j=1}^{N} \tilde{F}(\zeta, z_j). \tag{5.2}$$

In particular, if $N \geq 3$, and writing $z_0 = z_N$ only consecutive $\operatorname{supp} z_j$'s intersect each other, then

$$\tilde{F}(\zeta, \sum_{1}^{N} z_j) = \sum_{1}^{N} \tilde{F}(\zeta, z_{j-1} + z_j) - \sum_{1}^{N} \tilde{F}(\zeta, z_j).$$

Proof. It will suffice to verify (5.2) when $\tilde{F}(\zeta, z) = \tilde{E}(z)$ is homogeneous, in which case it follows by expressing both sides in terms of the polarization of $\tilde{E}$, and using Lemma 1.2a. The second formula follows from (5.2) by applying additivity to terms with non consecutive i, j.

Proof of Theorem 5.1. (a) Construction of the right inverse. By Theorem 1.1, for $F \in \mathfrak{F}$ we can choose $\tilde{F} \in \tilde{\mathfrak{F}}$, depending linearly on F, so that $F(\zeta, y) = \tilde{F}(\zeta, \dot{y})$. With $x \in L\mathbb{P}_1$ consider the differential form

$$F\Big(\zeta, \frac{1}{\zeta - x}\Big)d\zeta = \tilde{F}\Big(\zeta, \frac{\dot{x}}{(\zeta - x)^2}\Big)d\zeta, \tag{5.3}$$

holomorphic in $\mathbb{C}\backslash x(S^1)$. In fact, it is holomorphic at $\zeta = \infty$ as well, provided $\infty \notin x(S^1)$, since the coefficient of $d\zeta$ vanishes to second order at $\zeta = \infty$. This latter is easily verified

when $\tilde{F}(\zeta,z)=\zeta^\nu\tilde{E}(z)$ and $\tilde{E}$ is $(n+1)$–homogeneous, $\nu\le 2n$; in general it follows from the posthomogeneous expansion

$$\tilde{F}(\zeta,z)=\sum_0^\infty \tilde{F}_n(\zeta,z)=\sum_{n=0}^\infty\sum_{\nu=0}^{2n}\zeta^\nu\tilde{E}_{n\nu}(\zeta).$$

Hence, if $x\in L\mathbb{P}_1$ does not separate a and b, the integral

$$h_{ab}(x)=\int_a^b \tilde{F}\Big(\zeta,\ \frac{\dot{x}}{(\zeta-x)^2}\Big)d\zeta \tag{5.4}$$

is independent of the path joining a,b within $\mathbb{P}_1\backslash x(S^1)$, and defines a holomorphic function of a,b, and x.

We claim that h_{ab} can be continued to a cuspidal cocycle $\mathfrak{h}=(\mathfrak{h}_{ab})\in\mathfrak{H}$. First we prove a variant. Let $\sigma\in C^\infty(S^1)$ be supported in a closed arc $I\ne S^1$. Given finitely many $a,b,\ldots\in\mathbb{P}_1$, set

$$W_{ab\ldots}=\{x\in L\mathbb{P}_1\colon a,b,\ldots\notin x(I)\}\supset LU_{ab\ldots}.$$

We shall show that the integrals

$$\int_a^b \tilde{F}\Big(\zeta,\ \frac{\sigma\dot{x}}{(\zeta-x)^2}\Big)d\zeta,\qquad x \text{ does not separate } a,b, \tag{5.5}$$

can be continued to functions $\mathfrak{k}_{ab}(x)$ depending holomorphically on $a,b\in\mathbb{P}_1$, and $x\in W_{ab}$. The main point will be that, unlike $LU_{ab\ldots}$, the sets $W_{ab\ldots}$ are connected.

If $x_1\in W_{ab}$, construct a continuous curve $[0,1]\ni\tau\mapsto x_\tau\in W_{ab}$, $x_0=$ constant loop. Cover S^1 with open arcs $J_1,\ldots,J_N=J_0$, $N\ge 3$, so that only consecutive $\overline{J}_j$'s intersect, and no $x_\tau(\overline{J}_i\cup\overline{J}_j)$ separates a and b. Choose a C^∞ partition of unity $\{\rho_j\}$ subordinate to $\{J_j\}$. For x in a connected neighborhood $W\subset W_{ab}$ of $\{x_\tau\colon 0\le\tau\le 1\}$ define

$$\mathfrak{k}_{ab}(x)=\sum_1^N\int_a^b\tilde{F}\Big(\zeta,\ \frac{(\rho_{j-1}+\rho_j)\sigma\dot{x}}{(\zeta-x)^2}\Big)d\zeta-\sum_1^N\int_a^b\tilde{F}\Big(\zeta,\ \frac{\rho_j\sigma\dot{x}}{(\zeta-x)^2}\Big)d\zeta. \tag{5.6}$$

In the first sum we extend $(\rho_{j-1}+\rho_j)\sigma\dot{x}/(\zeta-x)^2$ to $S^1\backslash(J_{j-1}\cup J_j)$ by 0, and integrate along paths in $\mathbb{P}_1\backslash x(\overline{J}_{j-1}\cup\overline{J}_j)$; we interpret the second sum similarly. The neighborhood W is to be chosen so small that no $x(\overline{J}_i\cup\overline{J}_j)$ separates a and b when $x\in W$.

As above, the integrals in (5.6) are independent of the path, and define a holomorphic function in W. By Lemma 5.2, $\mathfrak{k}_{ab}$ agrees with (5.5) when x is near x_0. Furthermore, the germ of $\mathfrak{k}_{ab}$ at x_1 depends on the curve x_τ only through the choice of the ρ_j. In fact, it does not even depend on ρ_j: let $\mathfrak{k}'_{ab}$ be the function obtained if in (5.6) the ρ_j are replaced by another partition of unity ρ'_h. It will suffice to show that $\mathfrak{k}_{ab}=\mathfrak{k}'_{ab}$ under the additional assumption that each ρ'_h is supported in some J_j. In this case $\mathfrak{k}'_{ab}$ is holomorphic in W and agrees with $\mathfrak{k}_{ab}$ near x_0, hence on all of W.

Therefore, by varying the partition of unity ρ_j, we can use (5.6) to define $\mathfrak{k}_{ab}(x)$ depending holomorphically on $a, b \in \mathbb{P}_1$, and $x \in W_{ab}$. Also, $\mathfrak{k}_{ab} + \mathfrak{k}_{bc} = \mathfrak{k}_{ac}$ on W_{abc}, since this is so in a neighborhood of constant loops, and W_{abc} is connected.

Now, to obtain a continuation of h_{ab} in (5.4), construct a partition of unity $\sigma_1, \sigma_2, \sigma_3 \in C^\infty(S^1)$, so that $\operatorname{supp}(\sigma_i + \sigma_j) \neq S^1$ and $\bigcap_1^3 \operatorname{supp} \sigma_j = \emptyset$. Setting $\sigma_0 = \sigma_3$, in light of Lemma 5.2 we can rewrite (5.4)

$$h_{ab}(x) = \sum_1^3 \int_a^b \tilde{F}\Big(\zeta, \frac{(\sigma_{j-1} + \sigma_j)\dot{x}}{(\zeta - x)^2}\Big) d\zeta - \sum_1^3 \int_a^b \tilde{F}\Big(\zeta, \frac{\sigma_j \dot{x}}{(\zeta - x)^2}\Big) d\zeta,$$

and continue each term to LU_{ab}, as above. We obtain a holomorphic cocycle $\beta(F) = \mathfrak{h} = (\mathfrak{h}_{ab})$, with $\mathfrak{h}_{ab}$ depending holomorphically on a, b, and one easily checks that each $\mathfrak{h}_{ab}$ is cuspidal. Therefore $\beta(F) \in \mathfrak{H}$. Finally, $\alpha\beta(F)$ can be computed by considering $d_\zeta \mathfrak{h}_{a\zeta}(x)$ with a in the same component of $\mathbb{P}_1 \backslash x(S^1)$ as ζ, so that (5.4) gives

$$d_\zeta \mathfrak{h}_{a\zeta}(x) = d_\zeta h_{a\zeta}(x) = \tilde{F}\Big(\zeta, \frac{\dot{x}}{(\zeta - x)^2}\Big) d\zeta = F\Big(\zeta, \frac{1}{\zeta - x}\Big) d\zeta.$$

Thus $\alpha\beta(F) = F$ as needed.

(b) The kernel of α. Take an irreducible submodule of Ker α, spanned by a vector $\mathfrak{l}$ of lowest weight $-n \leq 0$. Since $F = \alpha(\mathfrak{l}) = 0$, (4.2) implies $\mathfrak{l}_{ab}(x) = 0$ if x does not separate a, b; hence, by analytic continuation, whenever $\operatorname{ind}_{ab} x = 0$. By the cocycle relation $\mathfrak{l}_{ac}(x) = \mathfrak{l}_{bc}(x)$ if $\operatorname{ind}_{ab} x = 0$, i.e., if $\operatorname{ind}_{ac} x = \operatorname{ind}_{bc} x$.

Consider the components of $LU_{0\infty}$

$$X_r = \{x \in LU_{0\infty} \colon \operatorname{ind}_{0\infty} x = r\}, \qquad r \in \mathbb{Z}.$$

Let

$$x_1(t) = e^{irt}, \quad y(t) = e^{2irt} + e^{3irt-4}. \tag{5.7}$$

We shall shortly show that whenever $x \in LU_{0\infty}$ is in a sufficiently small neighborhood of x_1, and $(\kappa, \lambda) \in \mathbb{C}^2 \backslash (0,0)$, then $z_{\kappa\lambda} = \kappa x + \lambda y \in X_r + \mathbb{C}$. It follows that with such x, y we can define $h(\kappa, \lambda) = \mathfrak{l}_{a\infty}(z_{\kappa\lambda})$, where a is chosen so that $\operatorname{ind}_{a\infty} z_{\kappa\lambda} = r$. Thus $h \in \mathcal{O}(\mathbb{C}^2 \backslash (0,0))$, and by Hartogs' theorem it extends to all of $\mathbb{C}^2$; also, it is homogeneous of degree $-n$. It follows that h is constant, indeed zero when $n > 0$. In all cases $\mathfrak{l}_{0\infty}(x) = h(1,0) = h(0,1)$ is independent of x. This being true for x in a nonempty open set, $\mathfrak{l}_{0\infty}$ is constant on X_r. It follows that $\mathfrak{l}_{a\infty}(x) = \mathfrak{l}_{0\infty}(x-a)$ is locally constant, and so is $\mathfrak{l}_{ab} = \mathfrak{l}_{a\infty} - \mathfrak{l}_{b\infty}$. Moreover, $\mathfrak{l}_{ab} = 0$ unless $n = 0$.

Suppose now $n = 0$, and let $\mathfrak{l}_{0\infty}|X_1 = l \in \mathbb{C}$. We have $\mathfrak{l}_{a\infty}(x) = \mathfrak{l}_{0\infty}(x - a) = l$ if $\operatorname{ind}_{a\infty} x = 1$. Choose a homeomorphic $x \in L\mathbb{C}$ and $a, b \in \mathbb{C} \backslash x(S^1)$ so that $\operatorname{ind}_{ab} x = 1$; say b is in the unbounded component. Then $\mathfrak{l}_{ab}(x) = \mathfrak{l}_{a\infty}(x) - \mathfrak{l}_{b\infty}(x) = l$, and the same will hold if x is slightly perturbed. It follows that $\mathfrak{l}_{ab}(x) = l$ whenever $\operatorname{ind}_{ab} x = 1$, and in

this case $\mathfrak{l}_{ba}(x) = -l$. Finally, with a generic $y \in LU_{ab}$ choose $a_0 = a, a_1, \dots, a_m = b$ in $\mathbb{P}_1 \backslash y(S^1)$ so that $\mathrm{ind}_{a_{j-1}a_j} y = \pm 1$. Then

$$\mathfrak{l}_{ab}(y) = \sum_1^m \mathfrak{l}_{a_{j-1}a_j}(y) = l \sum_1^m \mathrm{ind}_{a_{j-1}a_j} y = l \ \mathrm{ind}_{ab} y.$$

The upshot is that any irreducible submodule of Ker α is spanned by $\mathfrak{h}$ in (5.1), whence Ker α itself is spanned by $\mathfrak{h}$, as claimed.

We still owe the proof that $\kappa x + \lambda y \in X_r + \mathbb{C}$ unless $\kappa = \lambda = 0$, for x near x_1 and y given in (5.7). In fact, the general statement follows once we prove it for $r = 1$ and $x = x_1$, that we henceforward assume. If $|\kappa| \geq 2|\lambda|$ then $z_{\kappa\lambda} \in X_1$ by Rouché's theorem. Otherwise consider the polynomial

$$P(\zeta) = \kappa\zeta + \lambda(\zeta^2 + e^{-4}\zeta^3), \qquad \zeta \in \mathbb{C}.$$

For fixed $|\zeta| < 2$ the equation $P(\eta) = P(\zeta)$ has two solutions with $|\eta| < 5$, again by Rouché's theorem. One of the solutions is $\eta = \zeta$. Let $\eta = R(\zeta)$ be the other one, so that R is holomorphic. There are only finitely many ζ with $|\zeta| = |R(\zeta)| = 1$. Indeed, otherwise $|R(\zeta)| = 1$ would hold for all unimodular ζ, and by the reflection principle R would be rational. However, $P(R(\zeta)) = P(\zeta)$ cannot hold with rational $R(\zeta) \neq \zeta$. We conclude that $z_{\kappa\lambda}(S^1)$ has only finitely many self–intersection points.

Since $P(0) = 0$, $\mathrm{ind}_{0\infty} z_{\kappa\lambda} \geq 1$. Drag a point a from 0 to ∞ along a path that avoids multiple points of $z_{\kappa\lambda}(S^1)$. Each time we cross $z_{\kappa\lambda}(S^1)$, $\mathrm{ind}_{a\infty} z_{\kappa\lambda}$ changes by ± 1. It follows that $\mathrm{ind}_{a\infty} z_{\kappa\lambda} = 1$ for some a, which completes the proof.

For the space $L_{1,p}\mathbb{P}_1$ Theorems 2.1, 2.2, and the construction in Theorem 5.1 lead to explicit representations of elements of $\mathfrak{H}$. First there are the multiples of the cocycle (5.1), and then there is the complementary subspace $\beta(\mathfrak{F}) = \oplus_{n \leq p-1} \beta(\mathfrak{F}^n)$, see Theorem 2.2. According to Theorems 2.1, 2.2 elements of $\mathfrak{F}^n$ are of form

$$F(\zeta, y) = \sum_{\nu=0}^{2n} \zeta^\nu \langle \Phi_\nu, \dot{y}^{n+1} \rangle, \qquad \Phi_\nu \in L^{p/(n+1)}(S^1)^*.$$

Following the proof of Theorem 5.1, to compute $\mathfrak{h} = \beta(F)$ we set $\tilde{F}(\zeta, z) = \sum_\nu \zeta^\nu \langle \Phi_\nu, z^{n+1} \rangle$. The substitution $\zeta = \xi + c$ shows that

$$R_\nu(a, b, c) = \int_a^b \frac{\zeta^\nu d\zeta}{(\zeta - c)^{2n+2}}, \qquad 0 \leq \nu \leq 2n, \ c \in \mathbb{P}_1 \backslash \{a, b\},$$

are rational functions with poles at $c = a, b$, so that

$$\mathfrak{h}_{ab}(x) = \int_a^b \tilde{F}\Big(\zeta, \frac{\dot{x}}{(\zeta - x)^2}\Big) d\zeta = \sum_{\nu=0}^{2n} \langle \Phi_\nu, R_\nu(a, b, x) \dot{x}^{n+1} \rangle,$$

when x does not separate a, b. However, the right hand side makes sense for any $x \in LU_{ab}$ and, as one checks, defines $\mathfrak{h} = \beta(F)$. For example, if F, hence $\mathfrak{h}$ are of lowest weight, then $\Phi_\nu = 0$ for $\nu \geq 1$, and

$$\mathfrak{h}_{ab}(x) = \Big\langle \Phi_0,\ \frac{\dot{x}^{n+1}}{2n+1}\Big(\frac{1}{(x-a)^{2n+1}} - \frac{1}{(x-b)^{2n+1}}\Big)\Big\rangle. \tag{5.8}$$

Letting $n = 0$ and $\langle \Phi_0, z\rangle = \int_{S^1} z/2\pi i$, formula (5.8) recovers the locally constant cocycle (5.1) as well. Thus we proved

Theorem 5.3. *In the case of $W^{1,p}$ loop spaces, any lowest weight cocycle in the n'th isotypical subspace $\mathfrak{H}^n \subset \mathfrak{H}$ is of form (5.8) with (a unique) $\Phi_0 \in L^{p/(n+1)}(S^1)$, $0 \leq n \leq p-1$.*

6. Synthesis

In this last section we show how the results obtained by now imply the theorems of the Introduction. Theorems 0.1 and 0.2 follow from the isomorphism $H^{0,1}(L\mathbb{P}_1) \approx \mathfrak{H}$ of G–modules (Theorem 3.3) and from the isomorphism $\mathfrak{H} \approx \mathbb{C} \oplus \mathfrak{F}$, a consequence of Theorem 5.1. In particular, $H^{0,1}(L\mathbb{P}_1)^G \approx \mathbb{C} \oplus \mathfrak{F}^0$. The latter being isomorphic to the dual of $L^-\mathbb{C} = C^{k-1}(S^1)$, resp. $W^{k-1,p}(S^1)$ by Theorem 2.1, Theorem 0.3 also follows. Finally, Theorem 0.4 is a consequence of Theorems 2.2 and 2.1.

Seemingly we are done with all the proofs. However, Theorem 3.3 has not yet been proved for loop spaces $L_{1,p}\mathbb{P}_1$, $p < 3$, and we still have to revisit spaces of loops of low regularity. This will give us the opportunity to explicitly represent classes in $H^{0,1}(L_{1,p}\mathbb{P}_1)$, in fact, for all $p \in [1, \infty)$.

Generally, given a complex manifold M, $1 \leq p < \infty$, and a natural number $m \leq p$, consider the space $C^\infty_{0,q}((T^*M)^{\otimes m})$ of $(T^*M)^{\otimes m}$ valued $(0,q)$ forms on M. If ω is such a form, $v \in \oplus^q T^{0,1}_s M$, and $w \in T^{1,0}_s M$, we can pair $\omega(v) \in (T^*_s M)^{\otimes m}$ with $w^{\otimes m}$, to obtain what we shall denote $\omega(v, w^m) \in \mathbb{C}$. Write LM for the space of $W^{1,p}$ loops in M, and observe that the tangent space $T^{0,1}_x LM$ is naturally isomorphic to the space $W^{1,p}(x^*T^{0,1}M)$ of $W^{1,p}$ sections of the induced bundle $x^*T^{0,1}M \to S^1$ (see [L2, Proposition 2.2] in the case of C^k loops).

There is a bilinear map

$$I = I_q \colon L^{p/m}(S^1)^* \times C^\infty_{0,q}((T^*M)^{\otimes m}) \to C^\infty_{0,q}(LM),$$

obtained by the following Radon type transformation. If

$$(\Phi, \omega) \in L^{p/m}(S^1)^* \times C^\infty_{0,q}((T^*M)^{\otimes m}),$$

$x \in LM$, and $\xi \in \oplus^q T^{0,1}_x LM \approx \oplus^q W^{1,p}(x^*T^{0,1}M)$, then $\omega(\xi, \dot{x}^m) \in L^{p/m}(S^1)$. Define $I(\Phi, \omega) = f$ by

$$f(\xi) = \langle \Phi, \omega(\xi, \dot{x}^m)\rangle.$$

One verifies that $\overline{\partial} I(\Phi,\omega)=I(\Phi,\overline{\partial}\omega)$, whence I_q induces a bilinear map

$$L^{p/m}(S^1)^* \times H^{0,q}((T^*M)^{\otimes m}) \to H^{0,q}(LM).$$

Henceforward we take $M=\mathbb{P}_1,\ q=1,\ m=n+1$, and ω given on $\mathbb{C}$ by

$$\omega=\frac{-1}{2n+1}\,\frac{\overline{\zeta}^{2n}d\overline{\zeta}\otimes(d\zeta)^{n+1}}{(1+|\zeta|^{4n+2})^{(2n+2)/(2n+1)}},\qquad \zeta\in\mathbb{C},$$

so that $f=I_1(\Phi,\omega)$ is a closed form on $L\mathbb{P}_1$. Explicitly,

$$f(\xi)=\frac{-1}{2n+1}\left\langle\Phi,\ \frac{\xi\overline{x}^{2n}\dot{x}^{n+1}}{(1+|x|^{2n+2})^{(2n+2)/(2n+1)}}\right\rangle,\qquad \xi\in T_x^{0,1}L\mathbb{P}_1. \tag{6.1}$$

To compute its image in $\mathfrak{H}$ under the map of Theorem 3.3, let

$$\theta_a=\frac{1}{2n+1}\left(\frac{\zeta^{-2n-1}}{(1+|\zeta|^{4n+2})^{1/(2n+1)}}-\zeta^{-2n-1}+(\zeta-a)^{-2n-1}\right)(d\zeta)^{n+1}\qquad \text{on } U_a.$$

Thus $\overline{\partial}\theta_a=\omega|U_a$, and the cuspidal functions $u_a=I_0(\Phi,\theta_a)\in C^\infty(LU_a)$ solve $\overline{\partial}u_a=f|LU_a$. Hence the image of f in $\mathfrak{H}$ is

$$\mathfrak{h}_{ab}(x)=u_a(x)-u_b(x)=\left\langle\Phi,\ \frac{\dot{x}^{n+1}}{2n+1}\left(\frac{1}{(x-a)^{2n+1}}-\frac{1}{(x-b)^{2n+1}}\right)\right\rangle.$$

Comparing this with Theorem 5.3 we see that by associating with a lowest weight $\mathfrak{h}\in\mathfrak{H}^n$ the functional $\Phi=\Phi_0$ of (5.8), and then $f\in C^\infty_{0,1}(L\mathbb{P}_1)$ of (6.1), the image of f in $\mathfrak{H}$ will be $\mathfrak{h}$. In particular, the class $[f]\in H^{0,1}(L\mathbb{P}_1)$ is also of lowest weight $-n$. Therefore the linear map $\mathfrak{h}\mapsto[f]$, defined for $\mathfrak{h}\in\mathfrak{H}^n$ of lowest weight, can be extended to a G–morphism $\mathfrak{H}^n\to H^{0,1}(L\mathbb{P}_1)$, and then to a G–morphism $\bigoplus_{n\le p-1}\mathfrak{H}^n=\mathfrak{H}\to H^{0,1}(L\mathbb{P}_1)$, inverse to the morphism $H^{0,1}(L\mathbb{P}_1)\to\mathfrak{H}$ of Theorem 3.3. This completes the proof of Theorem 3.3, and now we are really done.

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